given by (8) must be in excess of the actual stored energy. This in turn means that the inductance approximation in (9) is always greater than the exact inductance, and that the capacitance approximation in (10) is always less than the exact capacitance.

The fact that the capacitance obtained with the uniform-current approximation is reasonably good for all R values means that (8) for the stored energy must also be a reasonably good approximation for all R values. This must mean that the gradient of (8) is very close to the actual energy gradient, i.e., that the force expression in (4) is a very good approximation and should be useful in structural design calculations for thin, parallel bus bars.

All of the results given here can also be obtained by starting with a uniform charge distribution. The resulting energy expression is identical to (8) with μI^2 replaced by Q^2/ϵ.

ACKNOWLEDGMENT

The author wishes to thank J. S. Ryberg who assisted with the calculations.

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An Accurate Determination of the Characteristic Impedance of the Coaxial System Consisting of a Square Concentric with a Circle

HENRY J. RIBLET

Bowman [1] has shown how the rectangular region in the w plane of Fig. 1, bounded by $0ABC$, may be mapped conformally into the trapezoidal region in the z plane of Fig. 1, bounded by $0ABC$, by means of the successive transformations

$$t = \mathrm{sn}^2\,(w,k) \tag{1}$$

$$u = \frac{k}{k'}\,(t-1)^{1/2} \tag{2}$$

$$\zeta = \left[\frac{2(k+ik')u}{(1+u)(k+ik'u)}\right]^{1/2} \tag{3}$$

$$z = \frac{1-i}{k+ik'}\left\{\int_0^{\zeta}\frac{d\zeta}{[(1-\zeta^2)(1-\lambda^2\zeta^2)]^{1/2}} - \int_0^{\zeta}\frac{d\zeta}{[(1-\zeta^2)(1-\lambda'^2\zeta^2)]^{1/2}}\right\} \tag{4}$$

Manuscript received October 7, 1974; revised February 14, 1975.
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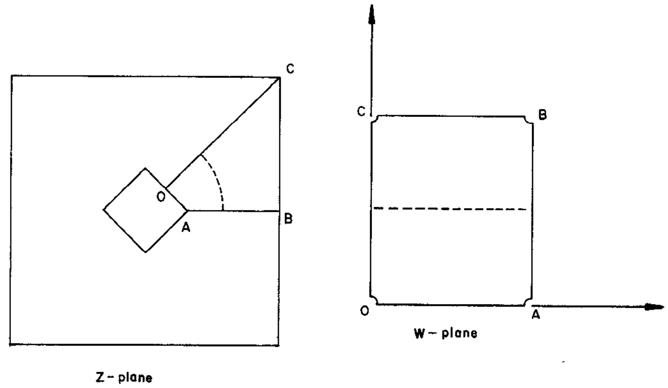

Fig. 1. Z and W coordinate planes.

where

$$\lambda = \frac{k^{1/2} + \dfrac{1+i}{(2)^{1/2}}\,(k')^{1/2}}{(2)^{1/2}(k+ik')^{1/2}}$$

and

$$\lambda' = \frac{(k)^{1/2} - \dfrac{1+i}{(2)^{1/2}}\,(k')^{1/2}}{(2)^{1/2}(k+ik')^{1/2}}.$$

Of course

$$\lambda^2 + \lambda'^2 = 1.$$

It is clear that the equipotentials in the w plane which are horizontal lines will map into portions of closed curves in the z plane which encircle the inner square and that for one of these equipotentials, points of intersection of the corresponding equipotential in the z plane with the line segments $0C$ and AB are equidistant from the center of the inner square. We are thus assured of the existence of an equipotential in the z plane which is equidistant from the center of the system at eight points spaced 45° apart. It is the object of this letter to show that this curve differs very little from a circle for the cases of greatest interest and that we have a means of determining with considerable accuracy the characteristic impedance of a coaxial line in which one of the conductors has a square cross section while the other has a circular cross section.

The problem of locating this equipotential in the z plane, and determining how much it differs from a circle depends on the evaluation of the incomplete elliptic integrals of the first kind in (4) in which the moduli are complex and the integration is performed, in general, along curves in the complex ζ plane. It turns out that the theory of second-order modular transformations [2], Landen transformations in complex form, is entirely adequate for the purpose, and that the integrals may be evaluated with the help of a digital computer with great accuracy.

Fig. 2 plots the values of the characteristic impedances for the cases where the outer conductor is a square or a circle as a function of the ratio of the circumference of the outer conductor to that of the inner conductor. For impedance values in the vicinity of 25 Ω, the curved equipotential is circular within ±0.2 percent, while at the other extreme, where the impedance values are approximately 100 Ω, the curved equipotential is circular within 2 parts in 10^7.

The reader should be reminded that k is the variable in terms of which the two characteristic impedances and the other variables of the problem may be expressed parametrically. It enters the analysis as the modulus of the Jacobian elliptic function, sn (w,k) by means of which the rectangle, $0ABC$, in the w plane is mapped into the upper-half t plane by (1). k varies from 0 to 1 and so controls the

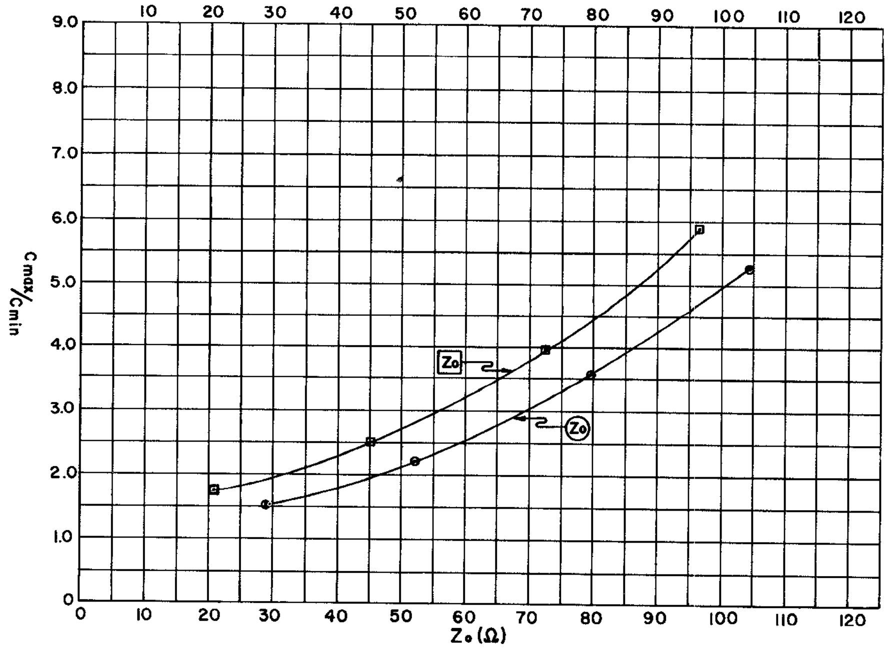

Fig. 2. Characteristic impedances.

relative shape of this rectangle which, in turn, determines whether the inner and outer conductors of the two coaxial systems will be close together or far apart. As $k \to 0$, side $AB \to \infty$, and both characteristic impedances $\to \infty$; while as $k \to 1$, side $0A \to \infty$ and both characteristic impedances $\to 0$. For the curves of Fig. 2, k varies from 0.005 to 0.65.

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Comments on "Transmission Line Impedance Matching Using the Smith Chart"

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For the recently discussed problem of R. M. Arnold[1] of finding the characteristic impedance and electrical length of a uniform, lossfree transmission line which transforms a given impedance into another given impedance, an alternate approach using an elementary auxiliary calculation (with no trial and error) has been given in [1].

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Manuscript received January 23, 1975; revised March 18, 1975.
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An Alternate Derivation and Procedure for Cristal's Transformation for Transmission-Line Networks

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Abstract—**This letter gives an alternate derivation and procedure for Cristal's transformation which transforms commensurate transmission-line networks with unit elements from the frequency variable λ to $1/\lambda$, where $\lambda = \tanh(\gamma L)$.**

Given a network with input impedance $Z(\lambda)$ composed of commensurate transmission-line elements including unit elements, we will now give a procedure for finding a network with input impedance $Z(1/\lambda)$ directly from the element values of the given circuit. This derivation is believed to be simpler, the method easier to apply and somewhat more general.

The method is based on the following theorem: If the network of Fig. 1(a) has input impedance $Z(\lambda)$ then the input impedance of the circuit of Fig. 1(b) is $Z(1/\lambda)$.

Proof: The input impedance of the network of Fig. 1(a) is given by

$$Z_A = Z_0 \frac{Z_L(\lambda) + \lambda Z_0}{Z_0 + \lambda Z_L(\lambda)}. \tag{1a}$$

Since the unit element in Fig. 1(b) is terminated by a load impedance $Z_0^2/Z_L(1/\lambda)$, the input impedance of the network of Fig. 1(b) is given by

$$Z_B = Z_0 \frac{Z_0^2/Z_L(1/\lambda) + \lambda Z_0}{Z_0 + \lambda[Z_0^2/Z_L(1/\lambda)]} = Z_0 \frac{Z_L(1/\lambda) + (1/\lambda)Z_0}{Z_0 + (1/\lambda)Z_L(1/\lambda)}. \tag{1b}$$

Comparison of (1a) and (1b) shows that $Z_B = Z_A(1/\lambda)$. Q.E.D. Using the above theorem we can easily transform ladder networks containing unit elements from the λ to the $1/\lambda$ plane. Consider as an example the network of Fig. 2(a) with input impedance $Z(\lambda)$. Using the above theorem we obtain the network of Fig. 2(b) with input impedance $Z(1/\lambda)$. Eliminating the impedance inverter then

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